MMS And APS Under Heterogeneous Cardinality Constraints

Arjun Aggarwal   Kyra Günlük   Ruta Mehta

Problem Definition and Abstract

We consider the problem of fair allocation of indivisible items to agents. In this problem, there is a set $M = \{1, 2, \ldots m\}$ of items, and each individual item must be given to exactly one of the agents in the set $N = \{1, 2, \ldots n\}$. Each agent assigns a 'cost' to each item, and the goal is to allocate the items fairly. An *allocation* $A = (A_1, A_2, \ldots A_n) \in \Pi_n(M)$ is an n-partition of M, with A_i being the bundle received by agent i. We compare an agents' *"cost"* for their allocated bundle to two known measures of fairness: Maximin Share (MMS) and Any Price Share (APS) as defined below. Our goal is to ensure a proportional factor of these measurements. We study both the "goods" and "chores" settings. In the "goods" settings, *"costs"* are non-negative, and indicate how much the agent values an item. The "chores" setting is identical, except *"costs"* are non-positive, which we instead replace by their negation, and refer to them as disutilites which represent how much an agent dislikes each item. In our problem, we impose an additional constraint that has not yet been explored: each agent has a unique cardinality condition on their assignment of items, represented by set $K = \langle k_1, k_2, \ldots k_n \rangle$. In the "goods" setting, the cardinality limits how many items each agent can have, so no agent gets too many, while in the "chores" case, the cardinality enforces how many items an agent must have, to ensure that no agent has too few chores.

Definitions for Goods

Goods Instance
An instance of the fair allocation problem under *heterogeneous cardinality constraints* is given by $I = \langle N, M, V, K \rangle$, where
$V = \langle v_1, v_2, \ldots v_n \rangle$ is the collection of the agents' valuation functions $v_i : 2^M \to \mathbb{R}_+$,
Final Valuation Let $\text{Top}_i^j(S)$ denote the set of $\max(|S|, j)$ most valuable goods for agent i in S. For any agent i, we define the *final valuation function* $f_i : 2^M \to \mathbb{R}_+$ as follows:

$$f_i(S) := \sum_{g \in \text{Top}_i^{k_i}(S)} v_i(g)$$

Maximin Share (MMS): The *maximin share* of i is defined as

$$\text{MMS}_i := \max_{A \in \Pi_n(M)} \min_{A_j \in A} f_i(A_j)$$

Any Price Share (APS): The *any price share* of i is defined as

$$\text{APS}_i := \min_{(p_1, p_2, \ldots p_m) \in P} \max_{S \subseteq M} \left\{ f_i(S) \mid \sum_{j \in S} p_j \leq \frac{1}{n} \right\}$$

Where $P = \{(p_1, p_2, ..., p_m) \mid p_j \geq 0 \, \forall j \in M, \sum_{j \in M} p_j = 1\}$ is the set of feasible item-price vectors.
Any Price Share (Dual Definition): The *any price share* of i is defined as

$$\text{APS}_i := \max z$$

Where z is subject to the following constraints:
1. $\sum_{T \subseteq M} \lambda_T = 1$ **2.** $\lambda_T \geq 0 \, \forall T \subseteq M$ **3.** $\lambda_T = 0 \, \forall T \subseteq M \text{ s.t. } f_i(T) < z$
4. $\sum_{T \subseteq M : j \in T} \lambda_T \leq \frac{1}{n} \, \forall j \in M$

Definitions for Chores

Chores Instance
An instance of the fair allocation problem under *heterogeneous cardinality constraints* is given by $I = \langle N, M, D, K \rangle$, where
$D = \langle d_1, d_2, \ldots d_n \rangle$ is the collection of the agents' disutility functions $d_i : 2^M \to \mathbb{R}_+$,
Minimax Share (MMS): The *minimax share* of i is defined as

$$\text{MMS}_i := \min_{A \in \Pi_n(M)} \max_{A_j \in A} d_i(A_j)$$

Any Price Share (APS): The *any price share* of i is defined as

$$APS_i := \max_{(r_1, r_2, \ldots r_m) \in R} \min_{S \subseteq M} \left\{ d_i(S) \mid \sum_{j \in S} r_j \geq \frac{1}{n}, |S| \geq k_i \right\}$$

Where $R = \{(r_1, r_2, \ldots r_m) \mid r_j \geq 0 \forall j \in M, \sum_{j \in M} r_j = 1\}$ is the set of feasible item-reward vectors.

Important Lemmas (Goods)

One-Good Reduction Lemma

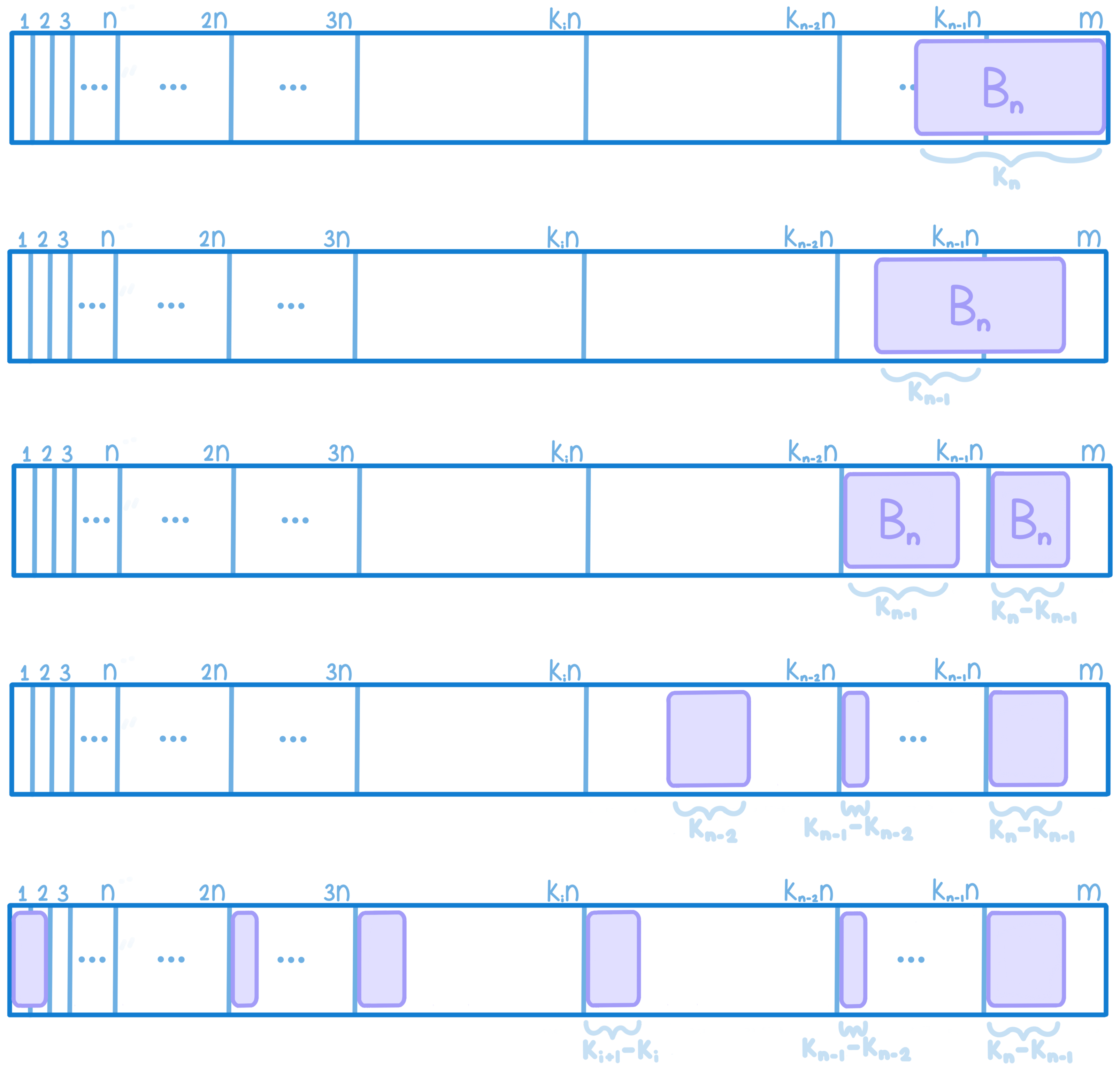
If a single item is given to an agent, and this item and agent are then removed from the instance, all remaining agents' APS remain the same or are increased.

This Lemma guarantees that any item valued at least $\frac{1}{2}$-APS by an agent can be given to that agent, and then both the item and agent can be removed from the instance.

1/2-MMS/APS Algorithm using Bag-Filling (Goods)

In this algorithm, we iterate over n and in each iteration we

- Create a bag with the k_{max} worst items remaining
- Begin sliding the 'window' of the bag over to include better items until an agent values the bag at least $\frac{1}{2}$.
- When the 'window' enters the top nk_i items for an agent i, the window splits from its original width, and only k_i of the window continues sliding. This ensures that at most k_i items are removed from agent i's top items.



Important Lemmas (Chores)

APS to Proportional Share Lemma

For any agent i, the APS value of the agent is at least their proportional share.

$$\text{APS}_i \geq \frac{1}{n} d_i(M)$$

This lemma provides that if item disutilities are scaled such that $d_i(M) = n$, then $\text{APS}_i \geq \frac{1}{n} d_i(M) = \frac{1}{n} n = 1$.

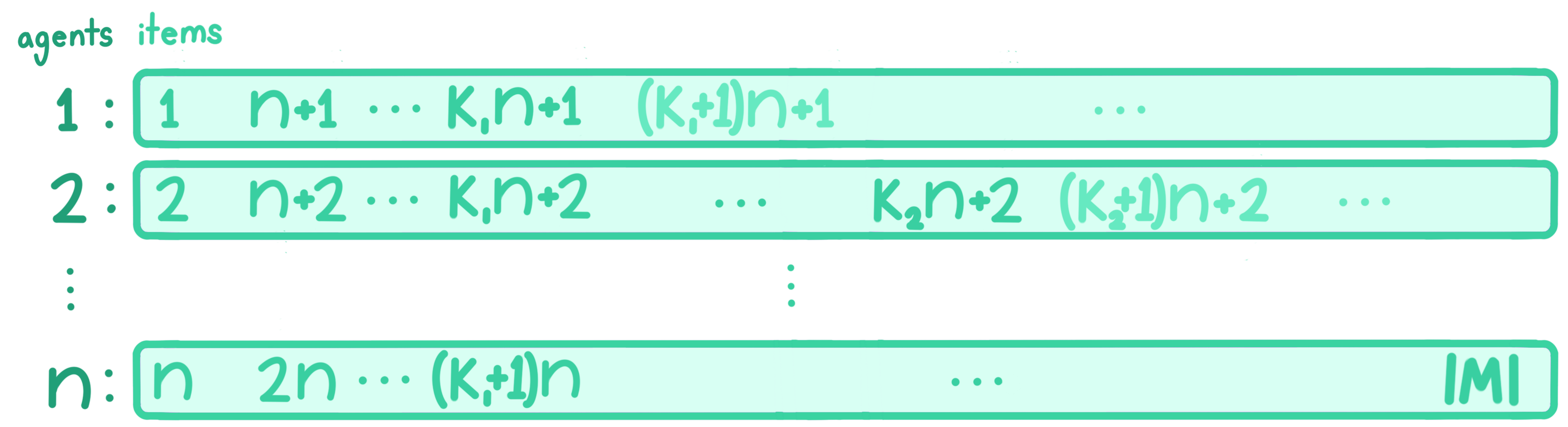
APS Disutility in Intervals Lemma

Let $S_k = \{kn - k + 1, kn - k \ldots kn + 1\}$ for any k such that $kn + 1 \leq |M|$

$$\text{APS}_i \geq d_i(S_k)$$

1/2 MMS/APS Algorithm Using Round Robin (Goods)

- First we will re-scale every agents valuations such that their value of the top nk_i items is equal to n, and consequently, their MMS and APS are at most 1.
- We then use the one good reduction to remove items that are valued at least 1/2 by some agent, so the remaining items are all valued less than 1/2.
- In every round, agents 1 through n will chose the best remaining item at their turn.



2 MMS/APS Algorithm Using Round Robin (Chores)

- First we will re-scale every agents disutilities such that their disutility of the items is equal to n, and consequently, their MMS and APS are at least 1.
- In every round, agents 1 through n will chose the most disutile remaining item at their turn.