Deterministically Approximating the Volume of a Hypercube Intersected with Halfspaces

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ABSTRACT

A deterministic approximation scheme for the volume of the hypercube $[0,1]^n$ intersected with a fixed number of halfspaces.
Happy 80th Birthday, Professor Frieze!

BACKGROUND

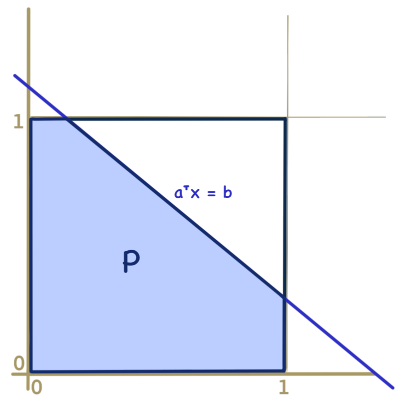
Theorem [Dyer, Frieze (88')]

Computing the volume of $P(A,b) = \{x : A^\top x \le b\}$ is #P-hard even when A is totally unimodular. In fact, it is #P-hard even for $P = [0,1]^n \cap \{x : a^\top x \le b\}$, the hypercube intersected with a single integral halfspace.

RESULTS

Single Halfspace

$$P = [0,1]^n \cap \{x : a^\top x \le b\}$$



Theorem: FPAS for vol(P), single halfspace

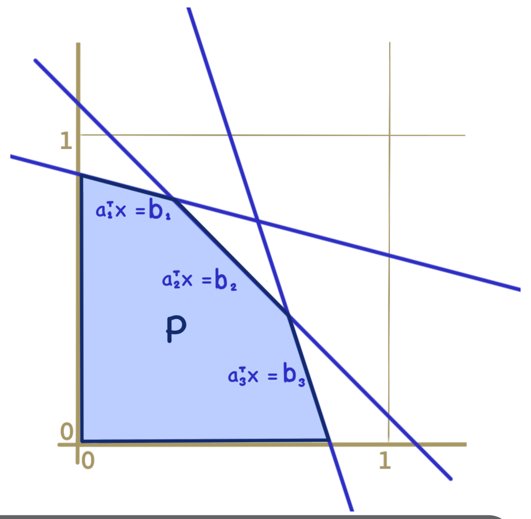
Given $a \in \mathbb{R}^n, b \in \mathbb{R}, \epsilon > 0, L \in \mathbb{Z}_+$, we can find Z' s.t.

$$vol(P) \le Z' \le (1+\epsilon)vol(P)$$

using $O(\frac{n^3 \log^4(n/\epsilon)}{\epsilon})$ arithmetic operations on $O(L + \log n)$-bit numbers, where L is the max bit size of any input number.

Multiple Halfspaces

$$P = [0,1]^n \cap \{x : A^\top x \le b\},$$
$$a_{ij}, b_i \ge 0 \; \forall i \in [k]$$

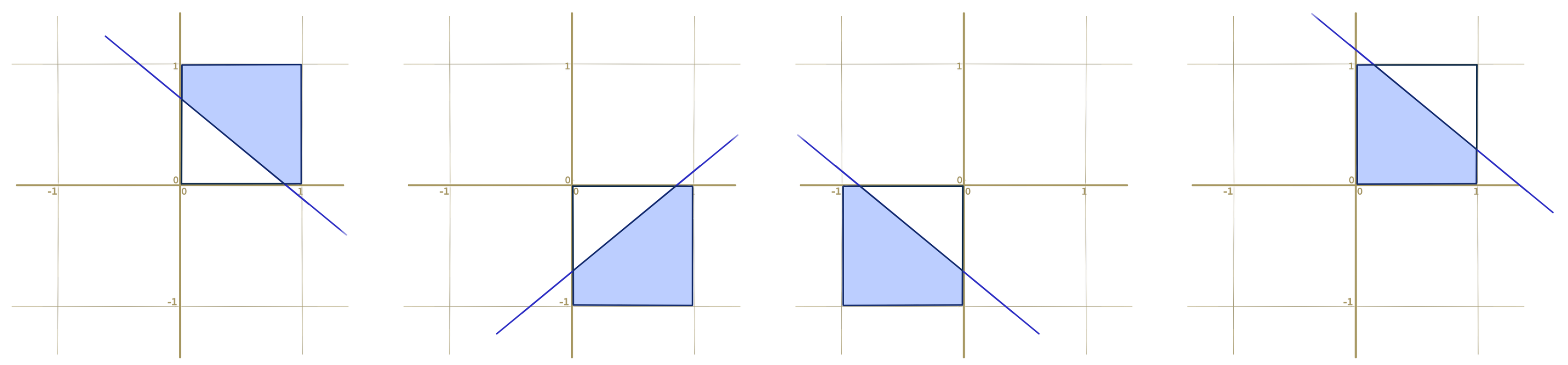


Theorem: FPAS for vol(P), multipel halfspaces

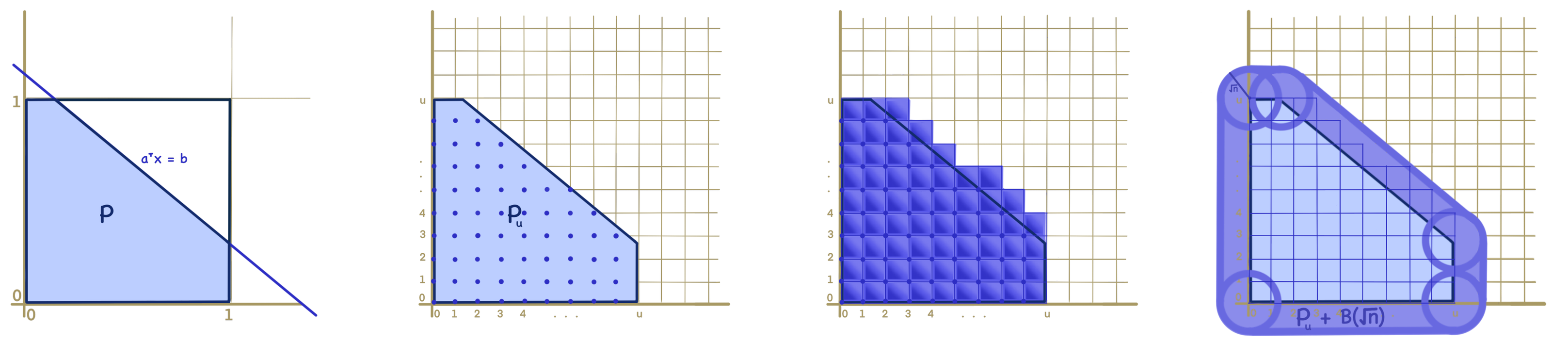
We can find Z' s.t. $vol(P) \le Z' \le (1+\epsilon)vol(P)$ in $n^{O(k^2)}(\log(\frac{n}{\epsilon})/\epsilon)^{O(k)}$ arithmetic operations on $O(L + \log n)$-bit numbers, where L is the max input bit size.

HYPERCUBE INTERSECTED WITH ONE HALFSPACE

Map to nonnegative constraint: Mirror P across axes using diagonal transformation matrices. Translate to positive orthant by increasing RHS by sum of absolute value of negative coefficients.



Scale: Dilate P by a factor of u in each dimension so that it lies in $[0,u]^n$ intersected with halfspace $a^\top x \le ub = c$. The volume of the dilated polytope P_u is exactly $\text{vol}(P_u) = u^n \cdot \text{vol}(P)$

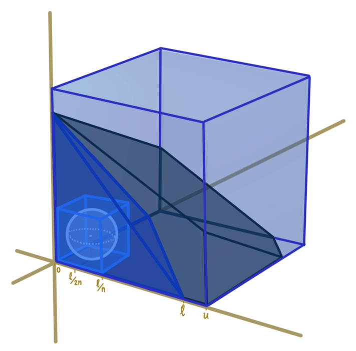


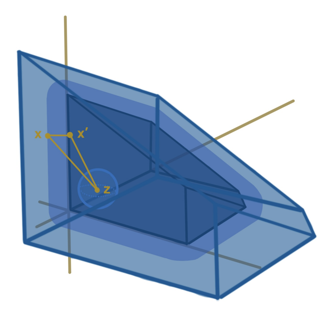
Using Integer Points to Estimate the Volume

For each integer point $\bar{x}$ in P_u, consider the unit cube $\{x : \bar{x} \le x \le \bar{x} + e\}$. P_u lies within the union of these cubes, C. Any point in cube $\{x : \bar{x} \le x \le \bar{x} + e\}$ lies within $\sqrt{n}$ of P_u, $P_u \subset C \subset P_u + B(\sqrt{n})$. vol$(C)$ = #cubes = #integer points in $[0, u-1]^n$ satisfying $a^\top x \le c$.

Sandwiching P

P_u contains a simplex of axes lengths at least ℓ, the smallest x_i intercept of our hyperplane, or u if this is smaller. The simplex contains a ball of radius $\frac{\ell}{2n}$.



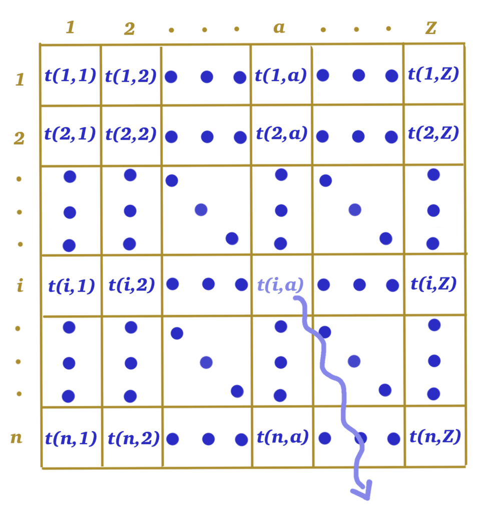


Next, $C \subset P_u + B(\sqrt{n}) \subset (1 + \frac{\sqrt{n}}{r})P_u$. Thus, $P_u \subseteq C \subseteq (1 + \frac{\sqrt{n}}{r})P_u$, and so C overestimates the volume of P_u by at most a factor of $(1 + \frac{\sqrt{n}}{r})^n$

APPROXIMATELY COUNTING KNAPSACK SOLUTIONS

Known Algorithm

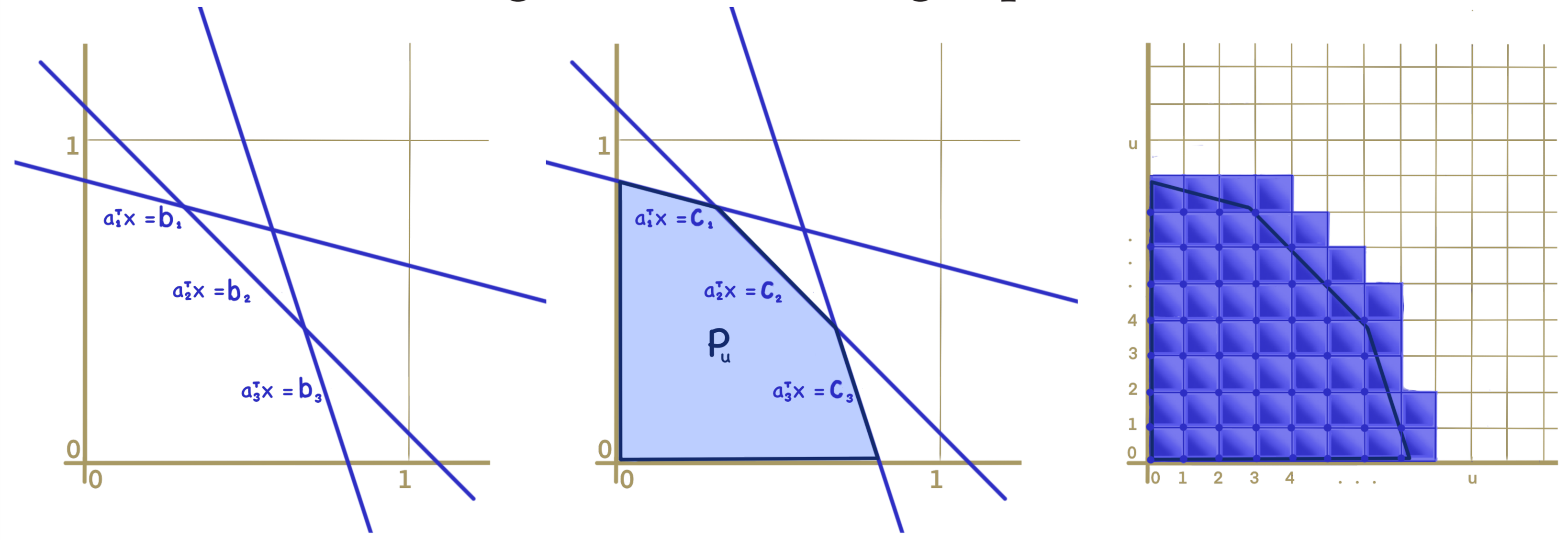
Counting integer solutions to $a^\top x \le c$ can be solved approximately with Dynamic Programming using $\tilde{O}(\frac{n^3 \log^3 u}{\epsilon'})$ operations [SVV10].



min C s.t. KNAP$(w_1, \dots, w_i, C)$ has at least a solutions

HYPERCUBE INTERSECTED WITH MULTIPLE HALFSPACES

Scale P to get $P_u = [0,u]^n \cap \{x : a_i^\top x \le ub_i = c_i \, \forall i\}$. Consider unit cubes extending from each integer point in P_u.

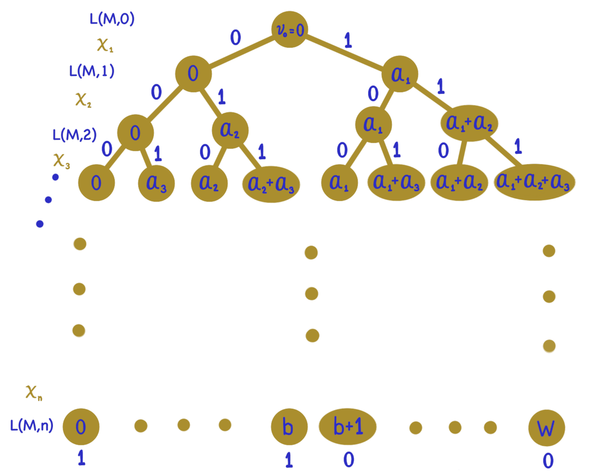


To approximate the number of integer solutions to the multi-dimensional knapsack problem, we can no longer use the DP approach, so we turn to a different algorithm using Read Once Branching Programs (ROBPs) [GKM10].

USING ROBPS TO COUNT KNAPSACK SOLUTIONS

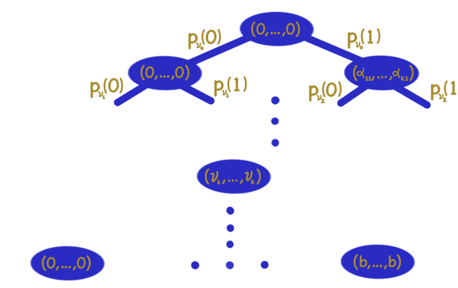
ROBP

For 0-1 knapsack constraint, ROBP has layers $0, 1, \dots, n$ with at most W vertices per layer. Vertices have edges $\{0,1\}$. Vertex v is a partial sum $\sum_{j=1}^{\ell} a_j x_j$ where x_j is edge prefix path from v_0 to v. Vertex in layer n is accepting (1) or rejecting (0). Probability of accepting from v_0 is the fraction of accepting solutions.



Small-Space Source:

Small-width ROBP with probability distributions on edges out of vertices.



Algorithm:

Select polynomial number of vertices from each constraint ROBP to approximate probability of solution sampled from small-space source being feasible in time $n^{O(k^2)}(\log U/\epsilon)^{O(k)} \log W$.

ONGOING WORK

Algorithm:

- Using Small-Space Sources to iteratively approximate volume of $[0,1]^n \cap \{x : a_1^\top x \le b_1\} \cap \dots \cap \{x : a_k^\top x \le b_k\}$.
- Using ROBPs to approximate volume of cube intersected with any read once monotone constraints i.e. $\sum_{j=1}^n a_{ij} x_j^{p_{ij}}$

REFERENCES

[DF88] M. E. Dyer and A. M. Frieze. On the complexity of computing the volume of a polyhedron. *SIAM J. Comput.*, 17(5):967–974, 1988.

[GKM10] Parikshit Gopalan, Adam R. Klivans, and Raghu Meka. Polynomial-time approximation schemes for knapsack and related counting problems using branching programs. *CoRR*, abs/1008.3187, 2010.

[SVV12] Daniel Stefankovic, Santosh S. Vempala, and Eric Vigoda. A deterministic polynomial-time approximation scheme for counting knapsack solutions. *SIAM J. Comput.*, 41(2):356–366, 2012.